

# The Geometric Dual of $a$ -Maximisation for Toric Sasaki–Einstein Manifolds

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**Abstract:** We show that the Reeb vector, and hence in particular the volume, of a Sasaki–Einstein metric on the base of a toric Calabi–Yau cone of complex dimension  $n$  may be computed by minimising a function  $Z$  on  $\mathbb{R}^n$  which depends only on the toric data that defines the singularity. In this way one can extract certain geometric information for a toric Sasaki–Einstein manifold without finding the metric explicitly. For complex dimension  $n = 3$  the Reeb vector and the volume correspond to the  $R$ -symmetry and the  $a$  central charge of the AdS/CFT dual superconformal field theory, respectively. We therefore interpret this extremal problem as the geometric dual of  $a$ -maximisation. We illustrate our results with some examples, including the  $Y^{p,q}$  singularities and the complex cone over the second del Pezzo surface.

## 1. Introduction

There has been considerable interest recently in Sasaki–Einstein geometry. Recall that a Sasaki–Einstein manifold  $Y$  is a Riemannian manifold of dimension  $(2n - 1)$  whose metric cone

$$ds^2(C(Y)) = dr^2 + r^2 ds^2(Y) \quad (1.1)$$

is Ricci–flat and Kähler. The recent interest has largely arisen due to a new construction of explicit inhomogeneous Sasaki–Einstein metrics in all dimensions [1–3]. In particular in dimension  $n = 3$  there is an infinite family of cohomogeneity one five–metrics, denoted  $Y^{p,q}$ , where  $q < p$  are positive integers [2]. The AdS/CFT correspondence [4] conjectures that for a Sasaki–Einstein five–manifold  $Y$ , type IIB string theory on  $AdS_5 \times Y$  with  $N$  units of self–dual five–form flux is dual to a four–dimensional  $\mathcal{N} = 1$  superconformal field theory [5–8]. This field theory may be thought of as arising from a stack of  $N$  D3–branes sitting at the apex  $r = 0$  of the corresponding Calabi–Yau cone (1.1). Following the results of [9], for the case  $Y = Y^{p,q}$  these field theories were

constructed in [10] thus furnishing a countably infinite set of AdS/CFT duals where both sides of the duality are known explicitly.

Recall that all Sasaki–Einstein manifolds  $Y$  have a canonically defined constant norm Killing vector field  $K$ , called the Reeb vector. In the case  $n = 3$  this is AdS/CFT dual to the R–symmetry of the dual superconformal field theory. The transverse geometry to the corresponding foliation of  $Y$  is always Kähler–Einstein of positive curvature. In the case that the leaves of the foliation are all compact one has a  $U(1)$  action on  $Y$ . If this action is free the Sasaki–Einstein manifold is said to be regular, and is the total space of a  $U(1)$  principle bundle over a positive curvature Kähler–Einstein manifold. More generally the  $U(1)$  action is only locally free and one instead has a  $U(1)$  orbibundle over a positive curvature Kähler–Einstein orbifold. Such structures are referred to as quasi–regular. If the generic orbits of  $K$  do not close there is only a transverse Kähler–Einstein structure and these are the irregular geometries.

In dimension five, regular Sasaki–Einstein metrics are classified completely [11]. This follows since the smooth four–dimensional Kähler–Einstein metrics with positive curvature on the base have been classified by Tian and Yau [12, 13]. These include the special cases  $\mathbb{C}P^2$  and  $S^2 \times S^2$ , with corresponding Sasaki–Einstein manifolds being the homogeneous manifolds  $S^5$  (or  $S^5/\mathbb{Z}_3$ ) and  $T^{1,1}$  (or  $T^{1,1}/\mathbb{Z}_2$ ), respectively. For the remaining metrics the base is a del Pezzo surface obtained by blowing up  $\mathbb{C}P^2$  at  $k$  generic points with  $3 \leq k \leq 8$  and, although proven to exist, the general metrics are not known in explicit form. In the last few years, starting with the work of Boyer and Galicki [14], quasi–regular Sasaki–Einstein metrics have been shown to exist on  $\#(S^2 \times S^3)$  with  $l = 1, \dots, 9$ . The irregular case is perhaps more interesting since so little is known about these geometries – the  $Y^{p,q}$  metrics [2] and their higher dimensional generalisations [3, 15, 16] are the very first examples. Indeed, these are counterexamples to the conjecture of Cheeger and Tian [17] that irregular Sasaki–Einstein manifolds do not exist.

For an irregular metric the closure of the orbits of  $K$  is at least a two–torus, meaning that the metric must possess at least a  $U(1) \times U(1)$  group of isometries. In this paper we restrict our attention to toric Sasaki–Einstein manifolds. By definition this means that the isometry group contains at least an  $n$ –torus. There are good mathematical and physical reasons for imposing toricity. On the mathematical side, as we shall see, the subject of toric Sasakian manifolds is simple enough that one can prove many general results without too much effort. On the physical side, for  $n = 3$ , a toric Sasaki–Einstein manifold is dual to a *toric quiver gauge theory*. These theories have a rich structure, but again are simple enough that one has considerable analytic control.

Given a Sasaki–Einstein five–manifold  $Y$ , the problem of constructing the dual field theory is in general a difficult one. However, provided the isometry group of  $Y$  is large enough one can typically make progress using a variety of physical and mathematical arguments. In particular, if  $Y$  is toric in principle<sup>1</sup> there is an algorithm which constructs the gauge theory from the toric data of the Calabi–Yau singularity [18, 19]. Thus in this case both the geometry and the gauge theory are specified by a set of combinatorial data. On physical grounds, this theory is expected to flow at low energies to a superconformal fixed point, and in particular the global symmetry group of this theory contains a canonical “ $U(1)_R$ ” factor, which is the R–symmetry. If this symmetry is correctly identified, many properties of the gauge theory may be determined. A general procedure that determines this symmetry is  $a$ –maximisation [20]. Roughly, one can define a function  $a$  on

<sup>1</sup> In practice this algorithm requires a computer, and even then one is limited to relatively small – in the sense of the toric diagram – singularities.

an appropriate space of admissible  $R$ -symmetries which depends only on the combinatorial data that specifies the quiver gauge theory, and thus in principle only on the toric data of the singularity. The local maximum of this function precisely determines the  $R$ -symmetry of the theory at its superconformal point. From the  $R$ -charges one can then use the AdS/CFT correspondence to compute the volume of the dual Sasaki–Einstein manifold, as well as the volumes of certain supersymmetric 3-dimensional submanifolds. Remarkable agreement was found for these two computations in the case of the  $Y^{2,1}$  metric [9], and the  $a$ -maximisation calculation [21] for the quiver gauge theory corresponding to the first del Pezzo surface [18]. The field theories for the remaining  $Y^{p,q}$  family were constructed in [10] and again perfect agreement was found for the two computations.

To summarise,  $a$ -maximisation and the AdS/CFT correspondence imply that the volumes of toric Sasaki–Einstein manifolds, as well as certain submanifolds, should somehow be extractable from the toric data of the Calabi–Yau singularity in a relatively simple manner, without actually finding the metric. In both the regular and quasi-regular cases this follows from the fact that, in these cases, one can view the Sasaki–Einstein manifold as a  $U(1)$  (orbi)-bundle over a Kähler–Einstein manifold (respectively orbifold), where the  $U(1)$  is generated by the Reeb vector. The problem of computing the volume, as well as the volumes of certain supersymmetric submanifolds, is then reduced to that of computing the volumes of the Kähler–Einstein base and its divisors, respectively, which is a purely topological question, see *e.g.* [22]. These are then clearly *rational* multiples of the volumes of the round five-sphere and three-sphere, respectively. However, in some sense the generic case is the irregular case and here one cannot reduce the computation to that of computing topological invariants. In this paper we show that one can determine the Reeb vector of any toric Sasaki–Einstein manifold in a simple way, without finding the metric, and from this one can compute the volumes referred to above. We therefore interpret this as being a geometric “dual” to  $a$ -maximisation.

## 2. Toric Sasakian Geometry

In this section we describe the Kähler geometry of toric varieties, focusing on the special case of a Kähler cone. The general formalism is due to Guillemin [23] and Abreu [24] and has been used recently in Donaldson’s work [25, 26] on constant scalar curvature metrics. Here we focus on the case where the Kähler toric variety is a cone over a real manifold, which by definition is a Sasakian manifold. The torus action fibres this Kähler cone over a rational polyhedral cone  $\mathcal{C} \subset \mathbb{R}^n$  via the moment map. Any toric Kähler metric may be written in terms of a symplectic potential, which is the Legendre transform of the Kähler potential, and in the special case of a cone we show that the moduli space of such symplectic potentials, for fixed toric variety, splits as

$$\mathcal{S} = \mathcal{C}_0^* \times \mathcal{H}(1), \quad (2.1)$$

where  $\mathcal{C}_0^*$ , the space of Reeb vectors, is the interior of the dual cone to  $\mathcal{C}$  and  $\mathcal{H}(1)$  is the space of smooth homogeneous degree one functions on  $\mathcal{C}$  (subject to a convexity condition). We also write down a Monge–Ampère equation in this formalism which imposes that the Sasakian metric is also Einstein. Regularity of a solution to this equation then imposes a condition on the Reeb vector  $K$ .

*Sasakian Geometry.* Let  $(X, \omega)$  be a Kähler cone of complex dimension  $n$ . This means that  $X = C(Y) \cong \mathbb{R}^+ \times Y$  has metric

$$ds^2(X) = dr^2 + r^2 ds^2(Y). \quad (2.2)$$

We take  $r > 0$  so that  $X$  is a smooth manifold which is incomplete at  $r = 0$ . The condition that this metric be Kähler is then equivalent to  $Y = X|_{r=1}$  being Sasakian – in fact this is probably the most useful definition of Sasakian. We then have

$$\mathcal{L}_{r\partial/\partial r}\omega = 2\omega \quad (2.3)$$

which says that the Kähler form  $\omega$  is homogeneous degree 2 under the Euler vector  $r\partial/\partial r$ . It follows that  $\omega$  is exact:

$$\omega = -\frac{1}{2}d(r^2\eta), \quad (2.4)$$

where  $\eta$  may be considered as a global one-form on  $Y = X|_{r=1}$ .

From this definition it is straightforward to show that the Reeb vector field

$$K \equiv \mathcal{I}\left(r\frac{\partial}{\partial r}\right) \quad (2.5)$$

is a Killing vector field, where  $\mathcal{I}$  denotes the complex structure on  $X$ .  $K$  is dual to the one-form  $r^2\eta$ , as follows simply from the above definitions. Thus equivalently we have

$$\eta = \mathcal{I}\left(\frac{dr}{r}\right). \quad (2.6)$$

In terms of the  $\partial$  operator on  $X$  we thus have

$$\eta = i(\partial - \bar{\partial})\log r \quad (2.7)$$

so that

$$d\eta = -2i\partial\bar{\partial}\log r. \quad (2.8)$$

Moreover one now computes that the Kähler form is simply

$$\omega = \frac{1}{2}i\partial\bar{\partial}r^2 \quad (2.9)$$

and thus we see that  $F \equiv r^2/4$  is a Kähler potential.

*Symplectic point of view.* We now impose in addition that  $(X, \omega)$  is toric. This means that the real torus  $\mathbb{T}^n$  acts effectively on  $X$ , preserving the Kähler form, which we regard as a symplectic form. Moreover one also requires that the torus action is integrable, meaning that one can introduce a moment map  $\mu : X \rightarrow \mathbb{R}^n$ . The moment map allows one to introduce symplectic coordinates on  $\mathbb{R}^n$ ,

$$y_i = -\frac{1}{2} < r^2 \eta, \frac{\partial}{\partial \phi_i} >, \quad (2.10)$$

where  $\partial/\partial \phi_i$  generate the  $\mathbb{T}^n$  action. Thus  $\phi_i$  are angular coordinates along the orbits of the torus action, with  $\phi_i \sim \phi_i + 2\pi$ . We may then use  $(y, \phi)$  as symplectic coordinates on  $X$ . Let us also assume<sup>2</sup> that  $X$  is of Reeb type. This means that there is some  $\zeta$  such that  $- < r^2 \eta, \zeta >$  is a strictly positive function on  $X$ . The moment map then exhibits the Kähler cone as a Lagrangian torus fibration over a strictly convex rational polyhedral cone  $\mathcal{C} \subset \mathbb{R}^n$  by forgetting the angular coordinates  $\phi_i$  [28]. This image is a subset of  $\mathbb{R}^n$  of the form

$$\mathcal{C} = \{y \in \mathbb{R}^n \mid l_a(y) \geq 0, a = 1, \dots, d\}, \quad (2.11)$$

where we have introduced the linear function

$$l_a(y) = (y, v_a) \quad (2.12)$$

with Euclidean metric  $(\cdot, \cdot)$ , and  $v_a$  are the inward pointing normal vectors to the  $d$  facets of the polyhedral cone. These normals are rational and hence one can normalise them to be primitive<sup>3</sup> elements of  $\mathbb{Z}^n$ . We also assume this set of vectors is minimal in the sense that removing any vector  $v_a$  in the definition (2.11) changes  $\mathcal{C}$ . The condition that  $\mathcal{C}$  be strictly convex is simply the condition that it is a cone over a convex polytope. There is an additional condition on the  $\{v_a\}$  for  $Y$  a smooth manifold, and the cone is then said to be *good* [27]. This may be defined as follows. Each face  $\mathcal{F} \subset \mathcal{C}$  may be realised uniquely as the intersection of some number of facets  $\{l_a(y) = 0\}$ . Denote by  $v_{a_1}, \dots, v_{a_N}$  the corresponding collection of normal vectors in  $\{v_a\}$ , where  $N$  is the codimension of  $\mathcal{F}$  – thus  $\{a_1, \dots, a_N\}$  is a subset of  $\{1, \dots, d\}$ . Then the cone is good if and only if

$$\left\{ \sum_{A=1}^N v_A v_{a_A} \mid v_A \in \mathbb{R} \right\} \cap \mathbb{Z}^n = \left\{ \sum_{A=1}^N v_A v_{a_A} \mid v_A \in \mathbb{Z} \right\} \quad (2.13)$$

for all faces  $\mathcal{F}$ .

The torus fibration is non-degenerate over the interior  $\mathcal{C}_0$  of  $\mathcal{C}$ . Thus the  $\mathbb{T}^n$  action is free on the corresponding subset  $X_0 = \mu^{-1}(\mathcal{C}_0)$  of  $X$ . The boundary  $\partial \mathcal{C}$  of the polyhedral cone then effectively describes  $X$  as a compactification of  $\mathcal{C}_0 \times \mathbb{T}^n$ . Specifically, the normal vector  $v_a \in \mathbb{Z}^n$  to a facet  $\{l_a(y) = 0\}$  determines a one-cycle in  $\mathbb{T}^n$  and this cycle collapses over the facet. Thus each facet corresponds to a toric symplectic subspace of  $X$  of real codimension two. Similarly lower-dimensional faces of the cone correspond to higher codimension toric symplectic subspaces. The condition that the cone is good then amounts to requiring that this compactification gives a cone over a smooth manifold  $Y$ .

<sup>2</sup> The symplectic toric cones that are *not* of Reeb type are rather uninteresting: they are either cones over  $S^2 \times S^1$ , cones over principle  $\mathbb{T}^3$  bundles over  $S^2$ , or cones over products  $\mathbb{T}^m \times S^{m+2j-1}$ ,  $m > 1$ ,  $j \geq 0$  [27].

<sup>3</sup> A vector  $v \in \mathbb{Z}^n$  is primitive if it cannot be written as  $mv'$  with  $v' \in \mathbb{Z}^n$  and  $\mathbb{Z} \ni m > 1$ .

The symplectic (Kähler) form is

$$\omega = dy_i \wedge d\phi_i, \quad (2.14)$$

where here and henceforth we adopt the Einstein summation convention for the indices  $\{i, j, k, \dots\}$ . As described in [24], any  $\mathbb{T}^n$ -invariant Kähler metric on  $X$  is then of the form

$$ds^2 = G_{ij} dy_i dy_j + G^{ij} d\phi_i d\phi_j, \quad (2.15)$$

where  $G^{ij}$  is the inverse matrix to  $G_{ij} = G_{ij}(y)$ . The almost complex structure is then clearly

$$\mathcal{I} = \begin{bmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{bmatrix} \quad (2.16)$$

in the basis  $(y, \phi)$  and it is straightforward to verify that integrability of  $\mathcal{I}$  requires  $G_{ij,k} = G_{ik,j}$  and hence

$$G_{ij} = G_{,ij} \equiv \frac{\partial^2 G}{\partial y_i \partial y_j} \quad (2.17)$$

for some strictly convex function  $G = G(y)$ . We refer to  $G$  as the symplectic potential for the Kähler metric. It should be clear that the metric (2.15) is a cone if and only if the matrix  $G_{ij}(y)$  is homogeneous degree  $-1$  in  $y$ .

*Complex point of view.* The introduction of the symplectic potential  $G(y)$  above may seem slightly mysterious, but in fact it is related to the more usual Kähler potential by Legendre transform. In fact the two viewpoints may be neatly summarised as follows. In the complex viewpoint one keeps the complex structure of  $X$  fixed and considers the Kähler form, and hence Kähler potential, to vary, whereas in the symplectic viewpoint one keeps the symplectic form fixed and varies the complex structure (2.16). Usually this latter approach is not particularly useful in Kähler geometry. However in toric Kähler geometry this formalism has already been used with great success, for example in Donaldson's work [25, 26] on constant scalar curvature metrics. This will also be the case for toric Sasakian metrics.

In the complex point of view one regards  $X$  as a complex algebraic variety coming equipped with a biholomorphic action of the complex torus  $\mathbb{T}_{\mathbb{C}}^n = (\mathbb{C}^*)^n$  which has a dense open orbit  $X_0$  which we identify with  $X_0$  above. We introduce standard complex coordinates  $w_i$  on  $\mathbb{C} \setminus \{0\}$ . The real torus  $\mathbb{T}^n \subset \mathbb{T}_{\mathbb{C}}^n$  then acts by translation in the imaginary direction for the log complex coordinates  $z_i = \log w_i = x_i + i\phi_i$ . The Kähler form  $\omega$  may then be written as

$$\omega = 2i \partial \bar{\partial} F, \quad (2.18)$$

where  $F = F(x)$  is the Kähler potential. Here we have again assumed that the metric is invariant under the  $\mathbb{T}^n$  symmetry. We also note that  $F(x)$  is a strictly convex function of the variables  $x$ . In these coordinates the metric is

$$ds^2 = F_{ij} dx_i dx_j + F_{ij} d\phi_i d\phi_j, \quad (2.19)$$

where

$$F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}. \quad (2.20)$$

It follows that

$$F_{ij}(x) = G^{ij}(y = \partial F / \partial x) \quad (2.21)$$

and the moment map is then clearly

$$\mu = y = \frac{\partial F}{\partial x} \quad (2.22)$$

by definition. It hence follows that the symplectic and Kähler potentials are related by Legendre transform

$$F(x) = \left( y_i \frac{\partial G}{\partial y_i} - G \right) (y = \partial F / \partial x). \quad (2.23)$$

*Delzant construction and the canonical metric.* Given a good strictly convex rational polyhedral cone  $\mathcal{C} \subset \mathbb{R}^n$  one can recover the original cone  $X$ , together with its symplectic structure, via symplectic reduction of  $\mathbb{C}^d$ . This follows from a generalisation [27] of Delzant’s theorem [29]. In fact  $X$  inherits a natural Kähler metric from Kähler reduction of the canonical metric on  $\mathbb{C}^d$ . The explicit formula for the symplectic potential of this metric for compact Kähler toric varieties was first given in a beautiful paper of Guillemin [23]. The case of singular varieties was studied recently in [30].

Denote by  $\Lambda \subset \mathbb{Z}^n$  the span of the normals  $\{v_a\}$  over  $\mathbb{Z}$ . This is a lattice of maximal rank. Consider the linear map

$$\begin{aligned} A : \mathbb{R}^d &\rightarrow \mathbb{R}^n, \\ e_a &\mapsto v_a, \end{aligned} \quad (2.24)$$

which maps each standard orthonormal basis vector  $e_a$  of  $\mathbb{R}^d$  to the primitive normal vector  $v_a$ . This induces a map of tori

$$\mathbb{T}^d \cong \mathbb{R}^d / 2\pi \mathbb{Z}^d \rightarrow \mathbb{R}^n / 2\pi \Lambda. \quad (2.25)$$

In general the kernel is  $\mathcal{A} \cong \mathbb{T}^{d-n} \times \Gamma$ , where  $\Gamma$  is a finite abelian group. Then  $X$  is given by the symplectic quotient

$$X = \mathbb{C}^d // \mathcal{A}. \quad (2.26)$$

One can describe this more explicitly as follows. One computes a primitive basis for the kernel of  $A$  over  $\mathbb{Z}$  by finding all solutions to

$$\sum_a Q_I^a v_a = 0 \quad (2.27)$$

for  $Q_I^a \in \mathbb{Z}$ , and such that for each  $I$  the  $Q_I^a$  have no common factor. The number of solutions, indexed by  $I$ , is  $d - n$  since  $A$  is surjective – this latter fact follows since  $\mathcal{C}$  is strictly convex. Then one has

$$X = \mathcal{K} / \mathbb{T}^{d-n} \times \Gamma \equiv \mathbb{C}^d // \mathcal{A} \quad (2.28)$$

with

$$\mathcal{K} \equiv \left\{ (Z_1, \dots, Z_d) \in \mathbb{C}^d \mid \sum_a Q_I^a |Z_a|^2 = 0 \right\} \subset \mathbb{C}^d, \quad (2.29)$$

where  $Z_a$  denote complex coordinates on  $\mathbb{C}^d$  and the charge matrix  $Q_I^a$  specifies the torus  $\mathbb{T}^{d-n} \subset \mathbb{T}^d$ . The quotient group  $\mathbb{T}^d/\mathcal{A} \cong \mathbb{T}^n$  then acts symplectically on  $X$  and by construction the image of the induced moment map  $\mu : X \rightarrow \mathbb{R}^n$  is the polyhedral cone  $\mathcal{C}$  that one began with. This is proven in [27].

Now  $X$  inherits a Kähler metric from the flat metric on  $\mathbb{C}^d$  via the reduction (2.28). Moreover from the latter equation we see that this induced metric is clearly invariant under homothetic rescaling of the  $\{Z_a\}$  and thus this metric will be a conical metric on  $X$ . There is an elegant expression for this metric, which in terms of the symplectic potential is given by [23]

$$G^{\text{can}}(y) = \frac{1}{2} \sum_a l_a(y) \log l_a(y). \quad (2.30)$$

We also note the following formulae:

$$\frac{\partial G^{\text{can}}}{\partial y_i} = \frac{1}{2} \sum_a [1 + \log l_a(y)] v_i^a, \quad (2.31)$$

$$G_{ij}^{\text{can}} = \frac{1}{2} \sum_a v_i^a v_j^a \frac{1}{l_a(y)}. \quad (2.32)$$

In particular note that  $G_{ij}^{\text{can}}$  is homogeneous degree  $-1$  which implies that the corresponding Kähler metric (2.15) is a cone. Also notice that  $G_{ij}^{\text{can}}$  has simple poles at each of the  $d$  facets  $l_a(y) = 0$ . This singular behaviour is required precisely so that the metric on  $\mathcal{C}_0 \times \mathbb{T}^n$  compactifies to a smooth<sup>4</sup> metric on  $X$ . As we shall see when we consider the Einstein condition for  $G(y)$ , the metric  $G_{ij}^{\text{can}}(y)$  is never Ricci-flat for  $d > n$ . The case  $d = n$  is the case that  $X$  is locally  $\mathbb{C}^n$ .

*The Reeb vector and moduli space of symplectic potentials.* Recall that on any Kähler cone  $(X, \omega)$  there is a canonically defined Killing vector field  $K$  defined by (2.5). In particular  $K$  has norm one at  $Y = \{r = 1\}$  and thus the orbits of  $K$  on  $Y$  define a foliation of  $Y$ . We refer to such a Sasakian structure as quasi-regular or irregular, depending on whether the generic orbits close or not, respectively. In the irregular case note that the isometry group is at least  $\mathbb{T}^m$ ,  $m \geq 2$ , with the orbits of the Killing vector filling out a dense subset of the orbits of the torus action. Indeed, the isometry group of a compact Riemannian manifold is always a compact Lie group. Hence the orbits of a Killing vector field define a one-parameter subgroup, the closure of which will always be an abelian subgroup and thus a torus. The dimension of the closure of the orbits,  $m$ , is called the rank.

It is also straightforward to show that the Reeb vector always lies in the centre of the Lie algebra of the automorphism group of  $Y$  – that is, the group of diffeomorphisms that

<sup>4</sup> When making such statements we always regard  $X$  as having its apex deleted.



preserve the Sasakian structure. To see this, suppose that the vector field  $V$  generates a symmetry of the Kähler cone. This means that  $V$  commutes with the Euler vector  $r\partial/\partial r$  and satisfies

$$\mathcal{L}_V \omega = 0, \quad \mathcal{L}_V \mathcal{I} = 0, \quad (2.33)$$

where  $\mathcal{L}$  denotes the Lie derivative. In particular  $V$  is an isometry of the metric<sup>5</sup>. We now compute

$$[V, K] = \mathcal{L}_V K = \mathcal{L}_V \left[ \mathcal{I} \left( r \frac{\partial}{\partial r} \right) \right] = 0. \quad (2.34)$$

Hence  $K$  commutes with  $V$  for all  $V$  and so  $K$  lies in the centre of the automorphism group.

For a toric Sasakian manifold we may write

$$K = b_i \frac{\partial}{\partial \phi_i} \quad (2.35)$$

and regard  $K$  as the vector  $b \in \mathbb{R}^n$ . Using

$$r \frac{\partial}{\partial r} = 2y_i \frac{\partial}{\partial y_i} \quad (2.36)$$

one easily computes that, for a given toric Sasakian manifold with symplectic potential  $G$ , we have

$$b_i = 2G_{ij}y_j. \quad (2.37)$$

It is straightforward to check that  $b$  is indeed a constant vector. For

$$\frac{\partial}{\partial y_k} b_i = 2y_j G_{ij,k} + 2G_{ik} = 2 \left( y_j \frac{\partial}{\partial y_j} \right) G_{ik} + 2G_{ik} = 0, \quad (2.38)$$

where we have used Euler's theorem and the fact that  $G_{ik}$  is homogeneous degree  $-1$ . For the canonical metric one easily computes

$$b^{\text{can}} = \sum_a v^a. \quad (2.39)$$

Suppose now that two different symplectic potentials  $G, G'$  have the same Reeb vector  $b \in \mathbb{R}^n$ . Defining  $g = G' - G$  we have

$$\left( y_j \frac{\partial}{\partial y_j} \right) \frac{\partial}{\partial y_i} g = 0 \quad (2.40)$$

so that  $g_{,i}$  is homogeneous degree 0 for each  $i$ . It follows that  $g \in \mathcal{H}(1)$  is homogeneous degree 1, up to a constant. To see this, note that (2.40) implies

$$\frac{\partial}{\partial y_i} \left[ \left( y_j \frac{\partial}{\partial y_j} \right) g - g \right] = 0, \quad (2.41)$$

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<sup>5</sup> The converse need not be true. The isometry group of the round  $S^5$  is  $SO(6)$  but the group which preserves a chosen complex structure is  $U(3)$ .

and hence

$$y_j \frac{\partial}{\partial y_j} g = g + t, \quad (2.42)$$

where  $t$  is a constant. The constant degree of freedom in  $G$  is clearly irrelevant. Indeed note that  $G'_{ij} = G_{ij}$  if and only if

$$g = \lambda_i y_i + t, \quad (2.43)$$

where  $\lambda_i, t$  are constants. Thus the symplectic potential should be thought of as being defined up to a linear function.

Conversely, if  $g = (G' - G) \in \mathcal{H}(1)$  then the two symplectic potentials  $G'$  and  $G$  define the same Reeb vector and indeed their Hessians are homogeneous degree  $-1$ .

Let us now define

$$G_b(y) = \frac{1}{2} l_b(y) \log l_b(y) - \frac{1}{2} l_\infty(y) \log l_\infty(y), \quad (2.44)$$

where

$$l_b(y) = (b, y) \quad (2.45)$$

and

$$l_\infty(y) = (b^{\text{can}}, y) = \sum_a (v_a, y). \quad (2.46)$$

Provided the plane  $l_b(y) = v > 0$  intersects the polyhedral cone  $\mathcal{C}$  to form a finite polytope, this function is a smooth function on  $\mathcal{C}$ . In fact this condition is that

$$(b, u_\alpha) > 0, \quad (2.47)$$

where the  $u_\alpha \in \mathbb{Z}^n$  are the generating edges of the cone  $\mathcal{C}$ . Indeed note that we may write

$$\mathcal{C} = \left\{ \sum_\alpha \lambda_\alpha u_\alpha \in \mathbb{R}^n \mid \lambda_\alpha \geq 0 \right\}. \quad (2.48)$$

This identifies  $\mathcal{C}^* = \{b \in \mathbb{R}^n \mid (b, u_\alpha) \geq 0\}$  as the dual cone to  $\mathcal{C}$ , which is also a convex rational polyhedral cone by Farkas' Theorem. Moreover,

$$2y_j \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_i} G_b = b_i - b_i^{\text{can}}, \quad (2.49)$$

and we may quite generally write any symplectic potential as

$$G = G^{\text{can}} + G_b + g, \quad (2.50)$$

where the Reeb vector for this potential is  $b$ , and  $g$  is a homogeneous degree one function. Since  $G^{\text{can}}$  already has the correct singular behaviour at the facets for the metric to compactify to a smooth metric on  $X$ , we simply require that  $g$  be smooth and  $b \in \mathcal{C}_0^*$  in order that this is also true for  $G$ . One also requires that  $G$  be strictly convex in order that the metric is positive definite.

We may summarise our results thus far as follows:

*The moduli space,  $\mathcal{S}$ , of symplectic potentials corresponding to smooth Sasakian metrics on some fixed toric Sasakian manifold  $Y$  can be naturally written as*

$$\mathcal{S} = \mathcal{C}_0^* \times \mathcal{H}(1), \quad (2.51)$$

*where  $b \in \mathcal{C}_0^* \subset \mathbb{R}^n$  labels the Reeb vector for the Sasakian structure, and  $g \in \mathcal{H}(1)$  is a smooth homogeneous degree one function on  $\mathcal{C}$ , such that  $G$  is strictly convex.*

*The Monge–Ampère equation.* Let  $F(x)$  denote the Kähler potential for a smooth metric on  $X$ , where recall that  $x_i$  are the real parts of complex coordinates on  $X$ . As is well known, the Ricci–form corresponding to  $F(x)$  is given by

$$\rho = -i\partial\bar{\partial} \log \det(F_{ij}). \quad (2.52)$$

Thus Ricci–flatness  $\rho = 0$  gives

$$\log \det(F_{ij}) = -2\gamma_i x_i + c, \quad (2.53)$$

where  $\gamma_i$  and  $c$  are constants, and we have noted that any  $\mathbb{T}^n$ -invariant pluri-harmonic function is necessarily of the form of the right-hand side. We may now take the Legendre transform of this equation to obtain

$$\det(G_{ij}) = \exp \left( 2\gamma_i \frac{\partial G}{\partial y_i} - c \right). \quad (2.54)$$

We will refer to this as the Monge–Ampère equation in symplectic coordinates.

Up until this point we have not imposed any Calabi–Yau condition on  $X$ . In particular if  $c_1(X)$  is non-zero one certainly cannot find a Ricci–flat metric. We thus henceforth take  $X$  to be a toric Gorenstein singularity. This means that, by an appropriate  $SL(n; \mathbb{Z})$  transformation, one can take the normal vectors for the polyhedral cone to be

$$v_a = (1, w_a) \quad (2.55)$$

for all  $a$ , where  $w_a \in \mathbb{Z}^{n-1}$ . In particular note this means that the charge vectors  $Q_I^a$  satisfy

$$\sum_a Q_I^a = 0 \quad (2.56)$$

for each  $I$  which in turn implies that  $c_1(X) = 0$ . The plot of the vectors  $w_a$  in  $\mathbb{Z}^{n-1}$  is usually called the toric diagram in the physics literature, at least in the most physically relevant case of  $n = 3$ .

Note that (2.54) implies that

$$-n = (b, \gamma). \quad (2.57)$$

This follows by taking the derivative of (2.54) along the Euler vector and the fact that the left-hand side is homogeneous degree  $-n$ . One now easily computes the right-hand side of the Monge–Ampère equation. Up to a normalisation factor we have

$$\det(G_{ij}) = \prod_a \left[ \frac{l_a(y)}{l_\infty(y)} \right]^{(v^a, \gamma)} [l_b(y)]^{-n} \exp \left( 2\gamma_i \frac{\partial g}{\partial y_i} \right). \quad (2.58)$$

Note that, since  $g \in \mathcal{H}(1)$ , the exponential is homogeneous degree 0, and hence the right-hand side is indeed homogeneous degree  $-n$ . However, in order that  $\det(G_{ij})$  has the correct singularity structure so that the corresponding Kähler metric is smooth, it must be of the form [24, 26]

$$\det(G_{ij}) = f(y) \prod_a [l_a(y)]^{-1}, \quad (2.59)$$

where  $f(y)$  is everywhere smooth on  $\mathcal{C}$  minus its apex. Thus we see that

$$(v_a, \gamma) = -1 \quad (2.60)$$

for all  $a$ . Clearly this is a very strong constraint and this is essentially where one sees  $c_1(X) = 0$ . For, if  $v_a = (1, w_a)$  then this is solved by taking

$$\gamma = (-1, 0, \dots, 0). \quad (2.61)$$

In particular from (2.57) we obtain

$$b_1 = n. \quad (2.62)$$

We conclude this subsection by deriving an expression for the holomorphic  $(n, 0)$ -form  $\Omega$  of the Ricci-flat metric on the Calabi-Yau cone. In complex coordinates, the  $(n, 0)$ -form may be written in the canonical form

$$\Omega = e^{i\alpha} (\det F_{ij})^{1/2} dz_1 \wedge \dots \wedge dz_n, \quad (2.63)$$

where  $\alpha$  is a phase which is fixed by requiring  $d\Omega = 0$ . Using Eq. (2.53) we obtain the following expression:

$$\Omega = e^{x_1 + i\phi_1} (dx_1 + i d\phi_1) \wedge \dots \wedge (dx_n + i d\phi_n). \quad (2.64)$$

Here we've set  $c = 0$ . Now, using (2.35), it is straightforward to derive the following:

$$\mathcal{L}_K \Omega = i n \Omega, \quad (2.65)$$

$$\mathcal{L}_{\frac{\partial}{\partial \phi_i}} \Omega = 0, \quad i = 2, \dots, n. \quad (2.66)$$

*The characteristic hyperplane and polytope.* Let us fix a toric Gorenstein singularity with polyhedral cone  $\mathcal{C} \subset \mathbb{R}^n$  and let  $G$  be a symplectic potential with Reeb vector  $b \in \mathcal{C}_0^*$ . The Reeb vector has norm one at  $Y = \{r = 1\}$ , which reads

$$1 = b_i b_j G^{ij} = 2b_i G_{jk} y_k G^{ij} = 2(b, y). \quad (2.67)$$

Thus the base of the cone  $Y$  at  $r = 1$  defines a hyperplane

$$\{y \in \mathbb{R}^n \mid (b, y) = \tfrac{1}{2}\} \quad (2.68)$$

with outward unit normal vector  $b/|b|$ . We call this the characteristic hyperplane for the Sasakian manifold [31]. Since  $b \in \mathcal{C}_0^*$  this hyperplane intersects  $\mathcal{C}$  to form a finite polytope  $\Delta = \Delta_b$ . We denote

$$H = \{y \in \mathbb{R}^n \mid (b, y) = \tfrac{1}{2}\} \cap \mathcal{C}. \quad (2.69)$$

Note that the Sasakian manifold  $Y$  is a  $\mathbb{T}^n$  fibration over  $H$ . Notice also that the Sasakian structure is quasi-regular if and only if  $b \in \mathbb{Q}^n$  is a rational point. One can interpret  $H$  as a Delzant–Lerman–Tolman polytope [32] if and only if the structure is quasi-regular and thus this polytope is rational.

Let us denote

$$X_1 = X|_{r \leq 1} \quad (2.70)$$

so that  $X_1$  is a finite cone over the base  $Y$ . Correspondingly the image

$$\mu(X_1) = \Delta = \Delta_b \quad (2.71)$$

under the moment map is the finite polytope  $\Delta$ , which depends on the Reeb vector  $b$ . The volume of  $X_1$  is

$$\text{vol}(X_1) = \int_0^1 dr r^{2n-1} \text{vol}(Y) = \frac{1}{2n} \text{vol}(Y). \quad (2.72)$$

On the other hand, since  $X$  is Kähler the volume form on  $X$  is simply  $\omega^n/n!$ . Integrating this over  $X_1$  one obtains

$$\int_{\mu^{-1}(\Delta)} \frac{1}{n!} \omega^n = \int_{\mu^{-1}(\Delta)} dy_1 \dots dy_n d\phi_1 \dots d\phi_n = (2\pi)^n \text{vol}(\Delta), \quad (2.73)$$

where  $\text{vol}(\Delta)$  is simply the Euclidean volume of the polytope  $\Delta$ . We thus have the simple result

$$\text{vol}(Y) = 2n(2\pi)^n \text{vol}(\Delta). \quad (2.74)$$

Note that this depends only on  $b$ , for fixed toric singularity, and not on the homogeneous degree one function  $g$ .

Let us now consider toric divisors in  $X$ . These are just the inverse images of the facets of  $\mathcal{C}$ . To see this, note that each facet is the reduction of  $\{Z_a = 0\} \subset \mathbb{C}^d$  in Delzant's construction, which clearly descends to a complex subspace of  $X$ . Thus each facet is the image under  $\mu$  of a toric divisor  $D_a$  in  $X$ . In particular the latter is calibrated by the form  $\omega^{n-1}/(n-1)!$ . A similar reasoning to the above then gives

$$\text{vol}(\Sigma_a) = (2n-2)(2\pi)^{n-1} \frac{1}{|v_a|} \text{vol}(\mathcal{F}_a), \quad (2.75)$$

where  $\mathcal{F}_a = \{l_a(y) = 0\} \cap \{r \leq 1\}$ ,  $v_a$  is the primitive normal vector, and  $\Sigma_a = \mu^{-1}(\mathcal{F}_a)|_{r=1}$  is the corresponding  $(2n-3)$ -submanifold of  $Y$ . Thus  $D_a = C(\Sigma_a)$ .

*To summarise, the volumes  $\text{vol}(Y)$  and  $\text{vol}(\Sigma_a)$  depend only on the Reeb vector  $b \in \mathcal{C}_0^*$  and not on the homogeneous degree one function  $g$ .*

This will be especially important when we consider Sasaki–Einstein metrics. In this case it is a very difficult problem to find  $b$  and the function  $g$  which satisfy the Monge–Ampère equation (2.58). However, as we shall demonstrate shortly, these two components essentially decouple from each other, and one can determine  $b$  for the Sasaki–Einstein metric independently of determining the function  $g$ .

A formula for the integrated Ricci scalar. According to [33] we have the following formula for the Ricci scalar<sup>6</sup>  $R_X$  of a toric Kähler metric on  $X$  in terms of the symplectic potential  $G$ :

$$R_X = -G^{ij}_{,ij} \equiv -G^{ij}_{,ij}. \quad (2.76)$$

Let us now integrate this formula over  $\Delta = \Delta_b$ . Using Stokes' theorem we have

$$\begin{aligned} \int_{\Delta} R_X dy_1 \dots dy_n &= - \int_{\Delta} G^{ij}_{,ij} dy_1 \dots dy_n \\ &= \sum_a \int_{\mathcal{F}_a} G^{ij}_i v_j^a \frac{1}{|v_a|} d\sigma - \int_H G^{ij}_i b_j \frac{1}{|b|} d\sigma, \end{aligned} \quad (2.77)$$

where  $d\sigma$  denotes the measure induced on a hyperplane. In fact the first term on the right-hand side of this equation is

$$\sum_a \frac{2}{|v_a|} \text{vol}(\mathcal{F}_a). \quad (2.78)$$

This is easily proved using the leading behaviour of  $G^{ij}$  near to the facets, which is universal in order that the metric be smooth. To see this, let us pick a facet, say  $\mathcal{F}_1$ , and take the normal vector to be  $v_1 = e_1 = (1, 0, \dots, 0)$ . Differentiating the relation

$$G^{ij} G_{jk} = \delta_k^i \quad (2.79)$$

and setting  $G = G^{\text{can}}$  we obtain

$$(G^{\text{can}})^{ij}_i \sum_a v_a^j v_a^k \frac{1}{l_a(y)} = (G^{\text{can}})^{ij} \sum_a v_a^i v_a^j v_a^k \frac{1}{l_a(y)^2}. \quad (2.80)$$

We now multiply this relation by  $l_1(y) = y_1$  and take the limit  $y_1 \rightarrow 0$ . One obtains

$$(G^{\text{can}})^{1i}_i(y_1 = 0) = \lim_{y_1 \rightarrow 0} \left[ (G^{\text{can}})^{11} \frac{1}{y_1} \right]. \quad (2.81)$$

Now taking the  $y_1 \rightarrow 0$  limit of (2.79) gives

$$\lim_{y_1 \rightarrow 0} \left[ (G^{\text{can}})^{11} \frac{1}{y_1} \right] = 2 \quad (2.82)$$

and thus we obtain

$$(G^{\text{can}})^{1i}_i(y_1 = 0) = 2. \quad (2.83)$$

The extension to general  $v_1$  now follows. It should also be clear from this argument that setting  $G = G^{\text{can}} + \tilde{G}$ , where  $\tilde{G}$  is smooth on the whole of  $\mathcal{C}$  gives the same result.

On the other hand, for the second term on the right-hand side of (2.77) we have

$$G^{ij}_i b_j = 2(G^{ij} G_{jk} y_k)_{,i} = 2y_{i,i} = 2n, \quad (2.84)$$

<sup>6</sup> We use a subscript  $X$  to distinguish this from the Ricci scalar of the Sasakian metric which will appear presently. Obviously the two are closely related.

and we thus obtain

$$\int_{\Delta} R_X dy_1 \dots dy_n = \sum_a \frac{2}{|v_a|} \text{vol}(\mathcal{F}_a) - \frac{2n}{|b|} \text{vol}(H). \quad (2.85)$$

However, we may now use the fact that [34]

$$\text{vol}(\Delta) = \frac{1}{2n|b|} \text{vol}(H). \quad (2.86)$$

This generalises the usual formula for the area of a triangle to higher dimensional polytopes. We give a proof of this in the next section. Together with the formulae (2.74), (2.75) we thus obtain

$$\begin{aligned} \int_{X_1} R_X dy_1 \dots dy_n &= (2\pi)^n \int_{\Delta} R_X dy_1 \dots dy_n \\ &= \frac{2\pi}{(n-1)} \sum_a \text{vol}(\Sigma_a) - 2n \text{vol}(Y). \end{aligned} \quad (2.87)$$

Note that for *compact* toric Kähler manifolds the last term is absent and, using another result from [34], one easily reproduces the formula in [33]. For our non-compact case of interest, we see that the integrated Ricci scalar of  $X$  is independent of  $g$ . Indeed, the right-hand side of (2.87) is manifestly only a function of the Reeb vector  $b$ .

We may now set  $R_X = 0$  for a Ricci-flat Kähler metric and we thus prove the relation

$$\pi \sum_a \text{vol}(\Sigma_a) = n(n-1) \text{vol}(Y). \quad (2.88)$$

Note that in the case of regular Sasaki–Einstein manifolds this formula in fact follows from a topological argument.

We conclude this section by deriving a relation valid for an arbitrary polytope in  $\mathbb{R}^n$ . The proof is again a simple application of Stokes' theorem. Consider the following form of Stokes' theorem:

$$\int_{\Delta} \nabla f dy_1 \dots dy_n = \int_{\partial \Delta} f v d\sigma, \quad (2.89)$$

where  $v$  is the outward-pointing normal vector to the boundary. Taking  $f$  to be the constant function, and using (2.86), gives immediately

$$\sum_a \frac{1}{|v_a|} \text{vol}(\mathcal{F}_a) v_a = 2n \text{vol}(\Delta) b, \quad (2.90)$$

where recall that the  $v_a$  are inward pointing, and  $b$  is outward pointing. As a first application of this result, consider the special case of a toric Gorenstein singularity, for which we can take the inward primitive normals to the facets to be of the form  $v_a = (1, w_a)$ . The first component of Eq. (2.90) then implies

$$\pi \sum_a \text{vol}(\Sigma_a) = b_1 (n-1) \text{vol}(Y), \quad (2.91)$$

where we have used (2.74) and (2.75) to pass from volumes of the polytope to  $Y$ . Comparing this with (2.88) we find that for Sasaki–Einstein metrics the component of the Reeb vector along the Calabi–Yau plane must be

$$b_1 = n. \quad (2.92)$$

Notice that the same result was obtained by studying regularity of the Monge–Ampère equation (2.54) on  $\mathcal{C}$ . A third derivation will be offered in the next section. Also note that this proves that the canonical metric  $G_{ij}^{\text{can}}$  is never Ricci–flat for  $d > n$ , since  $b_1^{\text{can}} = d$ . In the case  $d = n$  the metric on  $X$  is an orbifold of the flat metric on  $\mathbb{C}^n$ .

### 3. A Variational Principle for the Reeb Vector

In this section we derive a variational principle that determines the Reeb vector of a Sasaki–Einstein metric in terms of the toric data of a fixed toric Gorenstein singularity. The Reeb vector is the unique critical point of a function

$$Z : \mathcal{C}^* \rightarrow \mathbb{R} \quad (3.1)$$

which is closely related to the volume of the polytope  $\Delta$ . Existence and uniqueness of this local minimum is proven using a simple convexity argument. We examine the extremal function in detail in the case  $n = 3$  and determine the Reeb vector in a number of examples. In particular we correctly reproduce the Reeb vector and volumes for the explicit family of metrics  $Y^{p,q}$  and also examine the case of the suspended pinch point and the complex cone over the second del Pezzo surface. In the latter case no Sasaki–Einstein metric is known, or even known to exist. Nevertheless the dual field theories are known for all these singularities and the corresponding volumes can be computed in field theory using  $a$ –maximisation. For the second del Pezzo surface this computation was performed in [21], which corrected previous results in the literature. We find agreement with the computation obtained by extremising  $Z$ .

*The extremal function.* We begin with the Einstein–Hilbert action for a metric  $h$  on  $Y$ . This is given by a functional

$$S : \text{Met}(Y) \rightarrow \mathbb{R} \quad (3.2)$$

which explicitly is

$$S[h] = \int_Y (R_Y + 2(n-1)(3-2n)) \, d\mu_Y. \quad (3.3)$$

Here  $d\mu_Y$  is the usual measure on  $Y$  constructed from the metric  $h$  and  $R_Y = R_Y(h)$  is the Ricci scalar of  $h$ . The Euler–Lagrange equation for this action gives the Einstein equation

$$\text{Ric}_Y(h) = (2n-2)h. \quad (3.4)$$

This is equivalent to the metric cone

$$ds^2(X) = dr^2 + r^2h \quad (3.5)$$

being Ricci–flat.



We would like to interpret  $S$  as a functional on the space of *Sasakian* metrics on  $Y$ , and use the integral formula for the Ricci scalar of  $X$  derived in the previous section. The relationship between the Ricci scalar of the metric  $h$  on  $Y$  and the Ricci scalar of the cone  $X$  over  $Y$  is straightforward to derive:

$$R_X = \frac{1}{r^2} \left[ R_Y + (2n - 1) - (2n - 1)^2 \right]. \quad (3.6)$$

Integrating this over  $X_1$  gives

$$\int_{X_1} R_X = \frac{1}{2n - 2} \int_Y \left( R_Y + [(2n - 1) - (2n - 1)^2] \right) d\mu \quad (3.7)$$

and hence for a Sasakian metric  $h$  we compute

$$S[h] = 2(n - 1) \left[ \frac{2\pi}{n - 1} \sum_a \text{vol}(\Sigma_a) - 2n \text{vol}(Y) \right] + 4(n - 1) \text{vol}(Y) \quad (3.8)$$

giving

$$S = S[b] = 4\pi \sum_a \text{vol}(\Sigma_a) - 4(n - 1)^2 \text{vol}(Y). \quad (3.9)$$

Remarkably we see that the action depends only on  $b$ . Thus we may interpret  $S$  as a function

$$S : \mathcal{C}^* \rightarrow \mathbb{R}. \quad (3.10)$$

Moreover, Sasaki–Einstein metrics are critical points of this function. Thus we simply impose

$$\frac{\partial}{\partial b_i} S = 0 \quad (3.11)$$

which is a set of  $n$  algebraic equations for  $b$  in terms of only the toric data *i.e.* the normal vectors  $v_a$ .

We may write the function  $S$  more usefully as a function on the polytope  $\Delta$ :

$$Z[b] \equiv \frac{1}{4(n - 1)(2\pi)^n} S[b] = \sum_a \frac{1}{|v_a|} \text{vol}(\mathcal{F}_a) - 2n(n - 1) \text{vol}(\Delta). \quad (3.12)$$

Using (2.91) we can write this as

$$Z[b] = (b_1 - (n - 1)) 2n \text{vol}(\Delta_b). \quad (3.13)$$

Notice that  $Z[b]$  is then manifestly positive (negative) for  $b_1 > n - 1$  ( $b_1 < n - 1$ ).

It is interesting to take the derivative of  $S$  along the Euler vector on the dual cone  $\mathcal{C}^*$ :

$$b_i \frac{\partial}{\partial b_i} S = -2(n - 1)^2 \int_{X_1} R_X. \quad (3.14)$$

Thus we see that scalar flatness implies this component of the variational problem. Using (3.13) and the fact that  $\text{vol}(\Delta_b)$  is homogeneous degree  $-n$  in  $b$  we have

$$b_i \frac{\partial}{\partial b_i} Z = -2n(n-1)(b_1 - n) \text{vol}(\Delta), \quad (3.15)$$

and this in turn implies that  $b_1 = n$  for a critical point. Thus all critical points of  $Z$  lie on this plane in  $\mathcal{C}^*$ . Recall that this was also a necessary condition for a solution to the Monge–Ampère equation to correspond to a smooth metric on  $Y$ .

*Existence and uniqueness of an extremum.* We have shown that  $b_1 = n$  for all critical points of  $Z$ , and thus we may introduce a reduced function

$$\tilde{Z} = Z|_{b_1=n} = 2n \text{vol}(\Delta)|_{b_1=n}. \quad (3.16)$$

We must now set the variation of this to zero with respect to the remaining variables  $b_2, \dots, b_n$ .

There is a general formula for the volume of a convex polytope, and in principle one can carry out this extremisation explicitly. However, even in dimension  $n = 3$  the formula for  $\text{vol}(\Delta)$  can be quite unwieldy. We examine this general formula in more detail in the next subsection. In the current subsection we would instead like to prove that there is always a critical point of  $Z$  in  $\mathcal{C}^*$ , and moreover this critical point is unique and is a global minimum of  $\tilde{Z}$ . The critical point is therefore also the unique local minimum of  $Z$  – the global minimum is of course  $-\infty$ . The strategy is to show that  $\text{vol}(\Delta)$  is a strictly convex function on  $\mathcal{C}_0^*$ , and then use standard convexity arguments to argue for a unique critical point.

Let us first assume that  $\text{vol}(\Delta)$  is a strictly convex function of  $b$  on  $\mathcal{C}_0^*$ . It is simple to see that  $\text{vol}(\Delta)$  tends to  $+\infty$  everywhere on  $\partial\mathcal{C}^*$ . Geometrically this is the limit where the characteristic hyperplane  $H$  no longer intersects the polyhedral cone  $\mathcal{C}$  to form a finite polytope. Also note that  $\text{vol}(\Delta)$  is bounded below by zero and is continuous. Hence there must be some minimum of  $\tilde{Z}$  in the interior of the finite polytope in  $\mathcal{C}^*$  defined by  $b_1 = n$ . Moreover since  $\text{vol}(\Delta)$  is strictly convex there is a unique such critical point which is also a global minimum of  $\tilde{Z}$ , and we are done.

It remains then to prove that  $\text{vol}(\Delta)$  is strictly convex on  $\mathcal{C}_0^*$ . Our proof of this is remarkably simple. Let us write  $\Delta = \mathcal{C} \cap \{2(b, y) < 1\}$ , and set  $V(b) \equiv \text{vol}(\Delta)$ . Then

$$V = \int_{\Delta} dy_1 \dots dy_n = \int_{\mathcal{C}} \theta(1 - 2(b, y)) dy_1 \dots dy_n, \quad (3.17)$$

where we have introduced the Heaviside step function  $\theta(1 - 2(b, y))$ . Differentiating this with respect to  $b$  gives

$$\frac{\partial V}{\partial b_i} = - \int_H y_i \frac{1}{|b|} d\sigma, \quad (3.18)$$

where recall that the characteristic hyperplane  $H = \mathcal{C} \cap \{2(b, y) = 1\}$  and  $d\sigma$  is the usual measure on the hyperplane  $H \subset \mathbb{R}^n$ . Here we've simply used the fact that the derivative of the step function is a delta function. As a check on this formula, one can contract with  $b_i$  to obtain

$$b_i \frac{\partial V}{\partial b_i} = - \frac{1}{2|b|} \text{vol}(H). \quad (3.19)$$

However by Euler’s theorem the left-hand side is simply  $-nV$ , and hence we have proven the relation (2.86) that we used earlier.

We may now appeal to another result from reference [34], which again is straightforward to prove. Since  $y_i$  is homogeneous degree 1 we have  $(y_j \partial / \partial y_j) y_i = y_i$  and thus we compute

$$(n+1) \int_{\Delta} y_i \, dy_1 \dots dy_n = \int_{\Delta} \frac{\partial}{\partial y_j} (y_j y_i) \, dy_1 \dots dy_n = \frac{1}{2|b|} \int_H y_i d\sigma, \quad (3.20)$$

where in the last step we have used Stokes’ Theorem and the fact that on  $\partial\mathcal{C}$  we have  $(v_a, y) = 0$ . Thus

$$\frac{\partial V}{\partial b_i} = -2(n+1) \int_{\Delta} y_i \, dy_1 \dots dy_n. \quad (3.21)$$

Introducing a Heaviside function again and differentiating we thus obtain<sup>7</sup>

$$\frac{\partial^2 V}{\partial b_i \partial b_j} = \frac{2(n+1)}{|b|} \int_H y_i y_j d\sigma. \quad (3.22)$$

The integrand is now positive semi-definite, hence the Hessian of  $V$  is positive definite, and so  $V$  is strictly convex on  $\mathcal{C}_0^*$ .

*The extremal function in  $n = 3$  and examples.* The case of most physical interest is when the toric Calabi–Yau cone has complex dimension  $n = 3$ , and the corresponding Sasaki–Einstein manifold  $Y$  has real dimension five. Here we can give a simple formula for  $Z[b]$  and the volumes in terms of  $b$  and the toric data – namely the primitive normals  $v_a = (1, w_a)$  that define the polyhedral cone  $\mathcal{C}$ .

Denote by  $v_1, \dots, v_d$  the primitive normals, ordered in such a way that the corresponding facets are adjacent to each other, with  $v_{d+1} \equiv v_1$ . The volume of the  $a^{\text{th}}$  facet is then given by

$$\frac{1}{|v_a|} \text{vol}(\mathcal{F}_a) = \frac{1}{8} \frac{(v_{a-1}, v_a, v_{a+1})}{(b, v_{a-1}, v_a)(b, v_a, v_{a+1})}, \quad (3.23)$$

where  $(v, w, z)$  is the determinant of the  $3 \times 3$  matrix whose rows (or columns) are  $v, w$  and  $z$ , respectively. The volume of the polytope can for instance be obtained from the first component of (2.90)

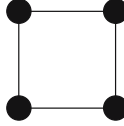
$$\text{vol}(\Delta_b) = \frac{1}{6b_1} \sum_a \frac{1}{|v_a|} \text{vol}(\mathcal{F}_a). \quad (3.24)$$

Clearly this is homogeneous degree  $-3$  in  $b$ . The volumes of the submanifolds  $\Sigma_a$  and the volume of  $Y$  are then determined using the formulae given earlier. Explicitly we have

$$\text{vol}(\Sigma_a) = 2\pi^2 \frac{(v_{a-1}, v_a, v_{a+1})}{(b, v_{a-1}, v_a)(b, v_a, v_{a+1})}, \quad (3.25)$$

$$\text{vol}(Y) = \frac{\pi^3}{b_1} \sum_a \frac{(v_{a-1}, v_a, v_{a+1})}{(b, v_{a-1}, v_a)(b, v_a, v_{a+1})}. \quad (3.26)$$

<sup>7</sup> It is straightforward to check this formula by brute force in dimension  $n = 2$ .



**Fig. 1.** Toric diagram for the conifold

*The conifold.* Let us start with the simplest and most familiar example of a toric non-orbifold singularity: the conifold. This is the Calabi–Yau cone over the homogeneous Sasaki–Einstein manifold  $T^{1,1}$ . The corresponding toric diagram is also well-known. A derivation of this starting from the conifold metric was presented in the Appendix of reference [9]. The inward pointing normals to the polyhedral cone in  $\mathbb{R}^3$  may be taken to be

$$v_1 = [1, 1, 1], \quad v_2 = [1, 0, 1], \quad v_3 = [1, 0, 0], \quad v_4 = [1, 1, 0]. \quad (3.27)$$

Projecting these onto the  $e_1 = 1$  plane one obtains the toric diagram in Fig. 1.

Notice that we have listed the normal vectors in the order of the facets of the polyhedral cone. The corresponding 3-submanifolds  $\Sigma_a$  are four copies of  $S^3$ . The extremal function is computed to be

$$Z[x, y, t] = \frac{(x - 2)x}{8yt(x - t)(x - y)}, \quad (3.28)$$

where here, and in the following examples, we set  $b = (x, y, t)$ . After imposing  $x = 3$  the remaining equations are then easily solved, and it turns out that there is a unique solution on  $\mathbb{R}^3$ . The extremising Reeb vector is

$$b_{\min} = \left(3, \frac{3}{2}, \frac{3}{2}\right). \quad (3.29)$$

One now easily computes

$$\text{vol}(\Sigma_a) = \frac{8}{9}\pi^2, \quad \frac{\pi}{6} \cdot 4 \cdot \frac{8}{9}\pi^2 = \frac{16}{27}\pi^3 = \text{vol}(T^{1,1}). \quad (3.30)$$

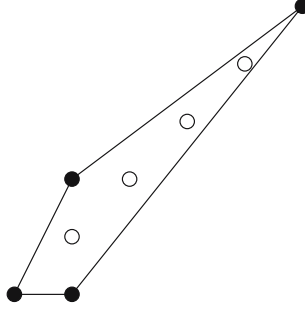
These results are in fact well-known in the physics literature.

*The  $Y^{p,q}$  toric singularities.* The  $Y^{p,q}$  toric singularities were determined in reference [9] by explicitly constructing the moment map for the  $\mathbb{T}^3$  action on the  $Y^{p,q}$  manifolds. The metrics on  $Y^{p,q}$  were constructed in references [1, 2]. The inward pointing normals to the four-faceted polyhedral cone may be taken to be

$$v_1 = [1, 0, 0], \quad v_2 = [1, p - q - 1, p - q], \quad v_3 = [1, p, p], \quad v_4 = [1, 1, 0]. \quad (3.31)$$

This corresponds to the basis of  $\mathbb{T}^3$  in which the toric diagrams were originally presented in reference [9]. Note that again we have listed the normals in the order of the facets of the polyhedral cone. In Fig. 2 we display, as an example, the case of  $Y^{5,3}$ . We compute the following function

$$\begin{aligned} Z[x, y, t] \\ = \frac{(x - 2)p(p(p - q)x + q(p - q)y + q(2 - p + q)t)}{8t(px - py + (p - 1)t)((p - q)y + (1 - p + q)t)(px + qy - (q + 1)t)}. \end{aligned} \quad (3.32)$$



**Fig. 2.** Toric diagram for  $Y^{5,3}$

Extremising this function is best left to Mathematica. Imposing  $x = 3$ , the remaining equations have four solutions on  $\mathbb{R}^3$ . However, only one lies within the dual cone  $\mathcal{C}^*$ , as must be the case from our earlier general analysis of the function  $Z$ . The final result is the following Reeb vector:

$$b_{\min} = \left( 3, \frac{1}{2}(3p - 3q + \ell^{-1}), \frac{1}{2}(3p - 3q + \ell^{-1}) \right), \quad (3.33)$$

where

$$\ell^{-1} = \frac{1}{q}(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}). \quad (3.34)$$

This is precisely the Reeb vector of the  $Y^{p,q}$  metrics [2, 9]. One then easily reproduces the total volume

$$\text{vol}(Y^{p,q}) = \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]} \pi^3 \quad (3.35)$$

and the volume of the supersymmetric submanifolds corresponding to the four facets [9, 10, 35], respectively.

*The suspended pinch point* The suspended pinch point (SPP) is a toric Gorenstein singularity where the five inward pointing normals to  $\mathcal{C}$  may be taken to be

$$\begin{aligned} v_0 &= [1, 0, 0], & v_1 &= [1, -1, 0], & v_2 &= [1, 0, 1], \\ v_3 &= [1, 1, 1], & v_4 &= [1, 1, 0]. \end{aligned} \quad (3.36)$$

Here we have also included the blow-up mode  $v_0$ .

Introducing the gauge-invariant monomials

$$u = Z_1 Z_4, \quad v = Z_2 Z_3, \quad w = Z_1^2 Z_2, \quad z = Z_3 Z_4^2, \quad (3.37)$$

we see that an equivalent algebraic description of the singularity is given by the hypersurface

$$wz = u^2 v \quad (3.38)$$

in  $\mathbb{C}^4$ . The boundary of this conical singularity is in fact singular. One can see this from the normal vectors as follows. Clearly  $\langle v_1, v_4 \rangle_{\mathbb{R}} \cap \mathbb{Z}^3$  is the sublattice  $\mathbb{Z}^2 \subset \mathbb{Z}^3$

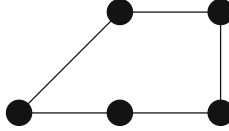


Fig. 3. Toric diagram for the SPP

spanned by  $e_1$  and  $e_2$ . However,  $\langle v_1, v_4 \rangle_{\mathbb{Z}}$  does not generate all of  $\mathbb{Z}^2$  – for example, one cannot generate the vector  $(1, 0, 0)$ . Thus the polyhedral cone is not good, in the sense of reference [27], and hence the boundary  $Y_{SPP}$  must be singular. Indeed, the two vectors  $v_1, v_4$  define an edge of the cone  $\mathcal{C}$ , and this edge does not satisfy the condition (2.13). In fact from the gauged linear sigma model it is easy to see [8] that  $Y_{SPP}$  is the cube root of the canonical circle bundle over the orbifold  $\mathbb{C}P^1_{[1,2]} \times \mathbb{C}P^1_{[1,2]}$ , where  $\mathbb{C}P^1_{[1,2]}$  is a weighted projective space – this is the symplectic quotient  $\mathbb{C}^2 // U(1)$ , where the  $U(1)$  has charges  $(1, 2)$ .

The function  $Z$  is given by

$$Z[x, y, t] = \frac{(x-2)(2x-t)}{8t(t-x)(t-x-y)(x-y)}. \quad (3.39)$$

This attains its local minimum at

$$b_{\min} = \left( 3, \frac{1}{2}(3 - \sqrt{3}), 3 - \sqrt{3} \right). \quad (3.40)$$

The volume of the corresponding Sasaki–Einstein metric<sup>8</sup> is then given by

$$\text{vol}(Y_{SPP}) = \frac{2}{9}\sqrt{3}\pi^3. \quad (3.41)$$

We also compute the following volumes:

$$\text{vol}(\Sigma_1) = \text{vol}(\Sigma_4) = \frac{2}{3}\pi^2, \quad \text{vol}(\Sigma_2) = \text{vol}(\Sigma_3) = \frac{2}{3}(-1 + \sqrt{3})\pi^2. \quad (3.42)$$

These results may be compared with the dual field theory calculations. The gauge theory for the SPP was obtained in reference [8] and it is straightforward to perform  $a$ -maximisation for this theory. Without entering into the details, we obtain the following function to maximise:

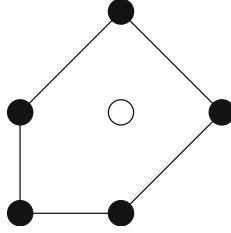
$$\begin{aligned} \frac{32}{9}a(x, y, z, t) = & 3 + (x-1)^3 + (y-1)^3 + (z-1)^3 + (t-1)^3 \\ & + (x+y-1)^3 + (1-x-y-z)^3 + (1-x-y-t)^3. \end{aligned} \quad (3.43)$$

Evaluating  $a$  at its local maximum gives<sup>9</sup>

$$a(Y_{SPP}) = \frac{3}{8}\sqrt{3}. \quad (3.44)$$

<sup>8</sup> This metric has recently been obtained in [36–38] as a member of an infinite family of toric Sasaki–Einstein metrics generalising  $Y^{P,q}$ . The volume indeed agrees with the value presented here.

<sup>9</sup> We suppress factors of  $N$ .



**Fig. 4.** Toric diagram for the complex cone over  $dP_2$

Using the AdS/CFT formula

$$a_Y = \frac{\pi^3}{4 \cdot \text{vol}(Y)}, \quad (3.45)$$

we therefore find perfect agreement with the geometrical result (3.41). It is quite remarkable that extremisation of the function  $Z$  in (3.39) and  $a$  in (3.43) are two completely equivalent problems.

*The complex cone over  $dP_2$ .* In the following we determine the Reeb vector for the Sasaki–Einstein metric<sup>10</sup> on the boundary of the complex cone over the second del Pezzo surface,  $dP_2$ . Recall that a del Pezzo surface  $dP_k$  is the blow-up of  $\mathbb{C}P^2$  at  $k$  generic points. Recall also that the first two del Pezzo surfaces do not admit Kähler–Einstein metrics [12, 13]. This fact follows straightforwardly from Matsushima’s theorem [39]. Thus the boundaries of the complex cones over  $dP_1$  and  $dP_2$  must be non-regular Sasaki–Einstein manifolds. In fact in [9] it was shown that  $Y^{2,1}$  is an irregular metric for the case of  $dP_1$ , while the metric for the case of  $dP_2$  remains unknown. Nevertheless, using our extremisation method one can determine the volume for this metric.

The five inward pointing normals may be taken to be

$$\begin{aligned} v_1 &= [1, 0, 0], & v_2 &= [1, 0, 1], & v_3 &= [1, 1, 2], \\ v_4 &= [1, 2, 1], & v_5 &= [1, 1, 0]. \end{aligned} \quad (3.46)$$

The extremal function in this basis is

$$Z[x, y, t] = \frac{(x-2)(-t^2 + 2t(x+y) + (3x-y)(x+y))}{8yt(t-x-y)(t+x-y)(t-3x+y)}. \quad (3.47)$$

The extremum that lies inside the dual cone is computed to be

$$b_{\min} = \left( 3, \frac{9}{16}(-1 + \sqrt{33}), \frac{9}{16}(-1 + \sqrt{33}) \right). \quad (3.48)$$

We may now compute the volume of the corresponding Sasaki–Einstein metric:

$$\text{vol}(Y_{dP_2}) = \frac{1}{486}(59 + 11\sqrt{33})\pi^3. \quad (3.49)$$

<sup>10</sup> Assuming that it exists.

This agrees with the value for this volume predicted by the authors of [21] using the purely field theoretic technique of  $a$ -maximisation together with the AdS/CFT formula (3.45). We also compute the following volumes:

$$\begin{aligned} \text{vol}(\Sigma_1) &= \frac{1}{81}(17 + \sqrt{33})\pi^2, & \text{vol}(\Sigma_2) &= \text{vol}(\Sigma_5) = \frac{1}{27}(1 + \sqrt{33})\pi^2, \\ \text{vol}(\Sigma_3) &= \text{vol}(\Sigma_4) = \frac{2}{81}(9 + \sqrt{33})\pi^2. \end{aligned} \quad (3.50)$$

It is then straightforward to match these with the R-charges of fields computed in reference [21].

#### 4. Discussion

In this paper we have shown that, for a given toric Calabi–Yau cone, the problem of determining the Reeb vector for the Sasaki–Einstein metric on the base of the cone is decoupled from that of finding the metric itself. The Reeb vector is determined by finding the unique critical point to the function

$$Z : \mathcal{C}^* \rightarrow \mathbb{R}. \quad (4.1)$$

It is then easy to see that this information is sufficient to compute the volume of the Sasaki–Einstein manifold, as well as the volumes of toric submanifolds which are complex divisors in the corresponding Calabi–Yau cone. For illustrative purposes, we have solved explicitly the extremal problem in a number of examples in complex dimension  $n = 3$ . One would also like to prove uniqueness and existence of a solution  $g \in \mathcal{H}(1)$  of the Monge–Ampère equation (2.58) to complete the analysis of toric Sasaki–Einstein manifolds, but we leave this for future work.

In the case of  $n = 3$  it is interesting to compare the geometrical results of this paper with  $a$ -maximisation in superconformal gauge theories in four dimensions. In order to do this, let us reformulate the extremal problem in the following way. A generic Reeb vector may be written

$$b = b_0 + \sum_{i=2}^n s_i b_i, \quad (4.2)$$

where  $b_0 = ne_1$ ,  $b_i = e_i$ ,  $i = 2, \dots, n$ , and  $s_i \in \mathbb{R}$ . The vector  $b_0$  is such that the  $(n, 0)$ -form  $\Omega$  of the Ricci-flat metric has charge  $n$  under the corresponding Killing vector field, whereas the  $b_i$  leave  $\Omega$  invariant. Indeed, recall that all critical points of  $Z$  necessarily lie on the plane  $(b, e_1) = n$ . The Reeb vector for the Sasaki–Einstein metric is then the unique global minimum of the reduced function  $\tilde{Z}$ , now regarded as a function of the parameters  $s_i$ . Moreover at the critical point,  $\tilde{Z}$  and  $Z$  are just the volume of the Sasaki–Einstein metric, up to a dimension-dependent factor.

Recall now that, starting from a toric Calabi–Yau singularity in complex dimension three, one can construct a four-dimensional supersymmetric quiver gauge theory arising from a stack of  $N$  D3-branes placed at the singularity, which is expected to flow at low energies to a non-trivial superconformal fixed point. The Higgs branch of this gauge theory is essentially the toric Calabi–Yau singularity.  $a$ -maximisation allows one to fix uniquely the exact R-symmetry of this theory at the infra-red fixed point. This may be formulated as follows. One first fixes a fiducial R-symmetry  $R_0$  which satisfies the



constraints imposed by anomaly cancellation. This R-charge is then allowed to mix with the set of global abelian non-R symmetries of the theory – by definition the supercharges are invariant under these symmetries. Thus the trial R-symmetry may be written as [20]

$$R = R_0 + \sum_I s_I F_I, \quad (4.3)$$

where  $F_I$  generate the group of abelian symmetries, and  $s_I \in \mathbb{R}$ . One can now define a function  $a$  which is a sum over a cubic function of the R-charges of fields in the theory, and is thus a function of the  $s_I$ . The exact R-symmetry of the theory at its conformal fixed point is uniquely determined by (locally) maximising this function  $a$  over the space of  $s_I$  [20]. Moreover, the value of  $a$  at the critical point is precisely the  $a$ -central charge of the gauge theory, which is inversely proportional to the volume of the dual Sasaki–Einstein manifold via the AdS/CFT formula (3.45).

Now, the AdS/CFT correspondence states that the subgroup of the isometry group of the Sasaki–Einstein manifold that commutes with the Reeb vector is precisely the set of flavour symmetries of the dual gauge theory. Recall that we showed that the Reeb vector cannot mix with the non-abelian part of the isometry group. In complex dimension  $n = 3$ , this is the geometrical realisation of the field theory statement that the R-symmetry does not mix with non-abelian factors of the global symmetry group of the gauge theory [20]. Therefore the minimisation of  $Z$  may always be performed over a space that is at most *two-dimensional*. Moreover, the  $b_i$ ,  $i = 1, 2$  precisely generate the  $U(1) \times U(1)$  isometry under which the  $(3, 0)$ -form is uncharged and are thus dual to flavour symmetries  $F_I$  in the gauge theory. In contrast, note that  $a$ -maximisation is generally performed over a larger parameter space, which includes the baryonic symmetries. However, the results here suggest that, for toric quiver gauge theories, it is possible to perform  $a$ -maximisation over a two-parameter space of flavour symmetries.

Notice that the problem of determining  $b_{\min}$  is reduced to finding the roots of polynomials whose degree generically increases with  $d$ , implying that the volumes, and hence also charges, of the dual theories are in general *algebraic numbers*. Although all theories considered in examples so far have been found to admit quadratic irrational charges, it is easy to see that more general algebraic numbers are expected as a result of maximising a cubic function of more than one variable. The precise relation between  $Z$  and  $a$  for a given toric singularity remains rather mysterious. It is clear that obtaining a 1–1 map between these two functions, and the details of the two extremal problems, would improve our understanding of some aspects of these superconformal field theories. Tackling this problem will require a better understanding of how the geometric data is translated into field theory quantities. One can anticipate that such quantities must be *invariants* with respect to the possible choices of toric phase or other field theory dualities.

Finally, we would like to emphasise that our results are valid in any dimension, while  $a$ -maximisation holds only for duals of five-dimensional Sasaki–Einstein geometries. However, the AdS/CFT correspondence predicts that  $AdS_4 \times Y_7$  geometries in M-theory, with  $Y_7$  a Sasaki–Einstein seven-manifold, are dual to *three-dimensional*  $\mathcal{N} = 2$  superconformal field theories. The results of this paper therefore suggest that there should exist some analogue of  $a$ -maximisation for three-dimensional theories as well. If true, the details of the argument should differ substantially from those used in reference [20] – in three dimensions there exist no anomalies to match. It will be very interesting to pursue this direction and explore the possibility that a field theoretic dual of  $Z$ -minimisation can be formulated for superconformal field theories in three dimensions.

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